FP1 JAN 11

1.

$$z = 5 - 3\mathrm{i}, \qquad w = 2 + 2\mathrm{i}$$

Express in the form $a + b\mathrm{i}$, where a and b are real constants,

(a) z^2,

(2)

(b) $\dfrac{z}{w}$.

(3)

a) $z^2 = (5-3i)^2 = 25 - 30i + 9i^2 = 16 - 30i$

b) $\dfrac{5-3i}{2+2i} \times \dfrac{2-2i}{2-2i} = \dfrac{10 - 16i + 6i^2}{4 - 4i^2} = \dfrac{4 - 16i}{8} = \dfrac{1}{2} - 2i$

2.

$$\mathbf{A} = \begin{pmatrix} 2 & 0 \\ 5 & 3 \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} -3 & -1 \\ 5 & 2 \end{pmatrix}$$

(a) Find **AB**.

(3)

Given that

$$\mathbf{C} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

(b) describe fully the geometrical transformation represented by **C**,

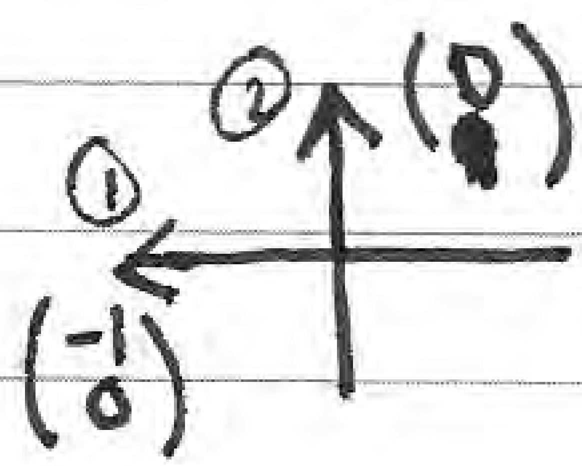
(2)

(c) write down $\mathbf{C}^{100}$.

(1)

a) $AB = \begin{pmatrix} 2 & 0 \\ 5 & 3 \end{pmatrix}\begin{pmatrix} -3 & -1 \\ 5 & 2 \end{pmatrix} = \begin{pmatrix} -6 & -2 \\ 0 & 1 \end{pmatrix}$

b) $C = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$



reflect through y-axis

c) $C^{100} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

3. $$f(x) = 5x^2 - 4x^{\frac{3}{2}} - 6, \quad x \geqslant 0$$

The root α of the equation $f(x) = 0$ lies in the interval $[1.6, 1.8]$.

(a) Use linear interpolation once on the interval $[1.6, 1.8]$ to find an approximation to α. Give your answer to 3 decimal places.

(4)

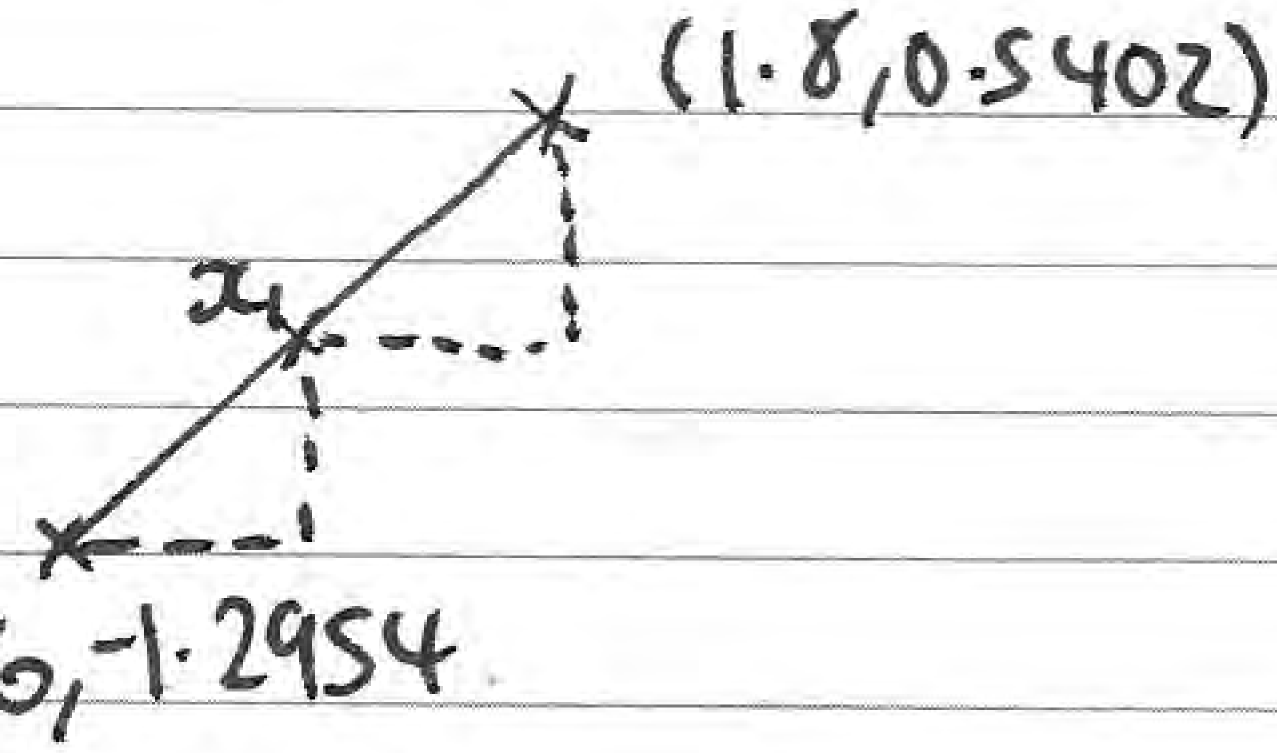
(b) Differentiate $f(x)$ to find $f'(x)$.

(2)

(c) Taking 1.7 as a first approximation to α, apply the Newton-Raphson process once to $f(x)$ to obtain a second approximation to α. Give your answer to 3 decimal places.

(4)

a) $f(1{\cdot}6) = -1{\cdot}2954$
$f(1{\cdot}8) = 0{\cdot}5402$



$$\frac{1{\cdot}8 - x}{x - 1{\cdot}6} = \frac{0{\cdot}5402}{1{\cdot}2954}$$

$1{\cdot}8 - x = 0{\cdot}41699x - 0{\cdot}66719$

$1{\cdot}41699x = 2{\cdot}46719 \Rightarrow x = \underline{1{\cdot}741}$

b) $f'(x) = 10x - 6x^{\frac{1}{2}}$

c) $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)} = 1{\cdot}7 - \dfrac{5(1{\cdot}7)^2 - 4(1{\cdot}7)^{\frac{3}{2}} - 6}{10(1{\cdot}7) - 6(1{\cdot}7)^{\frac{1}{2}}}$

$x_0 = 1{\cdot}7, \quad x_1 = 1{\cdot}745$

4. Given that 2 – 4i is a root of the equation

$$z^2 + pz + q = 0,$$

where p and q are real constants,

(a) write down the other root of the equation,

(1)

(b) find the value of p and the value of q.

(3)

a) $2+4i$

b) $\alpha = 2-4i$ $\quad \alpha+\beta = 4 = -p$ $\quad p = -4$

$\beta = 2+4i$ $\quad \alpha\beta = 4-16i^2 = 20 = q$ $\quad q = 20$

5. (a) Use the results for $\sum_{r=1}^{n} r$, $\sum_{r=1}^{n} r^2$ and $\sum_{r=1}^{n} r^3$, to prove that

$$\sum_{r=1}^{n} r(r+1)(r+5) = \frac{1}{4}n(n+1)(n+2)(n+7)$$

for all positive integers n.

(5)

(b) Hence, or otherwise, find the value of

$$\sum_{r=20}^{50} r(r+1)(r+5)$$

(2)

a) $\sum r = \frac{1}{2}n(n+1)$

$\sum r^2 = \frac{1}{6}n(n+1)(2n+1)$

$\sum r^3 = \frac{1}{4}n^2(n+1)^2$

$r(r+1)(r+5)$

$(r^2+r)(r+5)$

(r^3+6r^2+5r)

$= \frac{1}{4}n^2(n+1)^2 + \frac{1}{4}\times 4n(n+1)(2n+1) + \frac{1}{4}\times 10n(n+1)$

$= \frac{1}{4}n(n+1)[n(n+1)+4(2n+1)+10]$

$= \frac{1}{4}n(n+1)[n^2+9n+14] = \frac{1}{4}n(n+1)(n+2)(n+7)$ #

b) $\left[\frac{1}{4}n(n+1)(n+2)(n+7)\right]_{19}^{50} = \left(\frac{1}{4}(50)(51)(52)(57) - \left(\frac{1}{4}(19)(20)(21)(26)\right)\right)$

$= 1837680$

6.

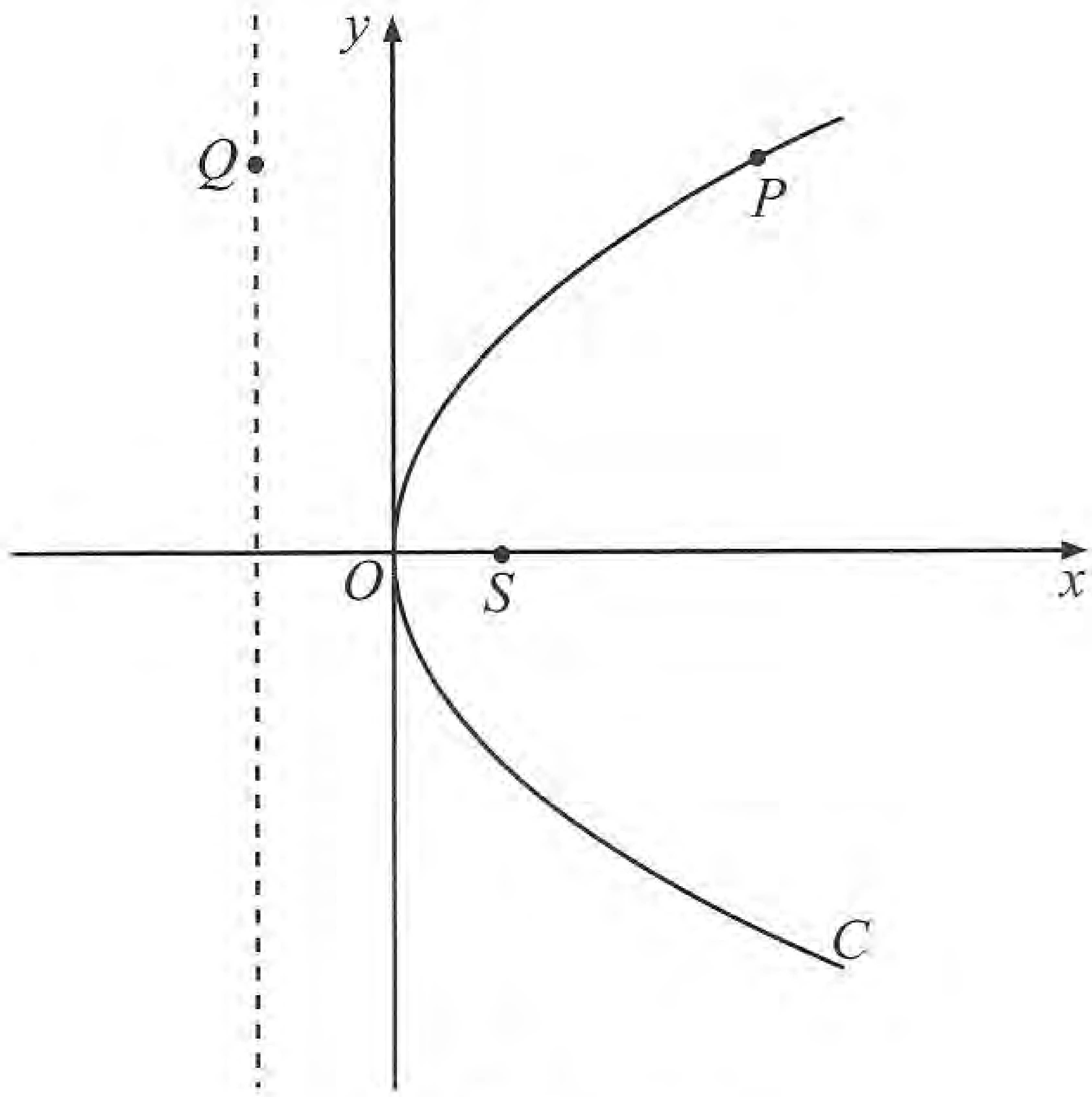


Figure 1

Figure 1 shows a sketch of the parabola C with equation $y^2 = 36x$. $= 4ax \quad a = 9$
The point S is the focus of C.

(a) Find the coordinates of S. $(9, 0)$
(1)

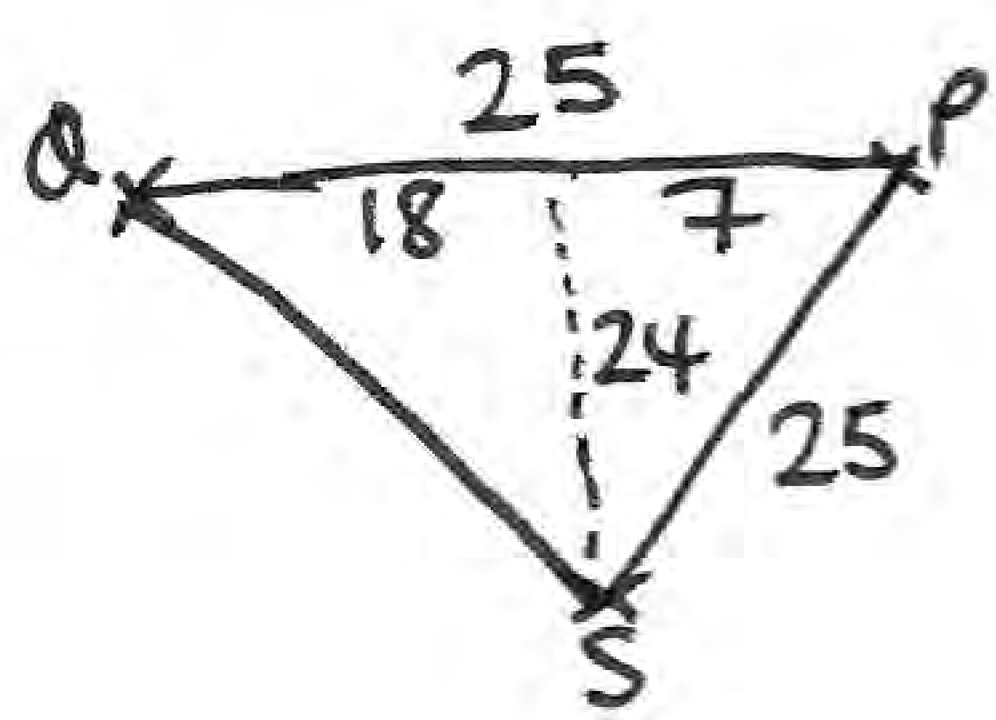
(b) Write down the equation of the directrix of C. $x + 9 = 0 \quad x = -9$
(1)

Figure 1 shows the point P which lies on C, where $y > 0$, and the point Q which lies on the directrix of C. The line segment QP is parallel to the x-axis.

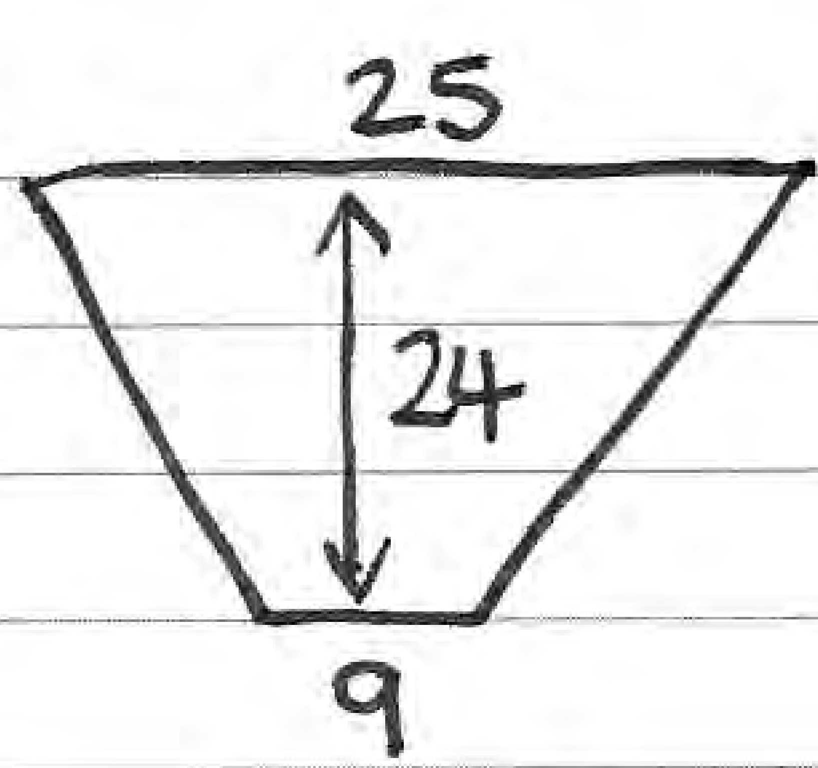
Given that the distance PS is 25,

(c) write down the distance QP, $= 25$
(1)

(d) find the coordinates of P, $(16, 24)$
(3)



(e) find the area of the trapezium $OSPQ$.
(2)



$\frac{34 \times 24}{2} = \underline{408}$

7. $$z = -24 - 7\text{i}$$

(a) Show z on an Argand diagram. **(1)**

(b) Calculate arg z, giving your answer in radians to 2 decimal places. **(2)**

It is given that

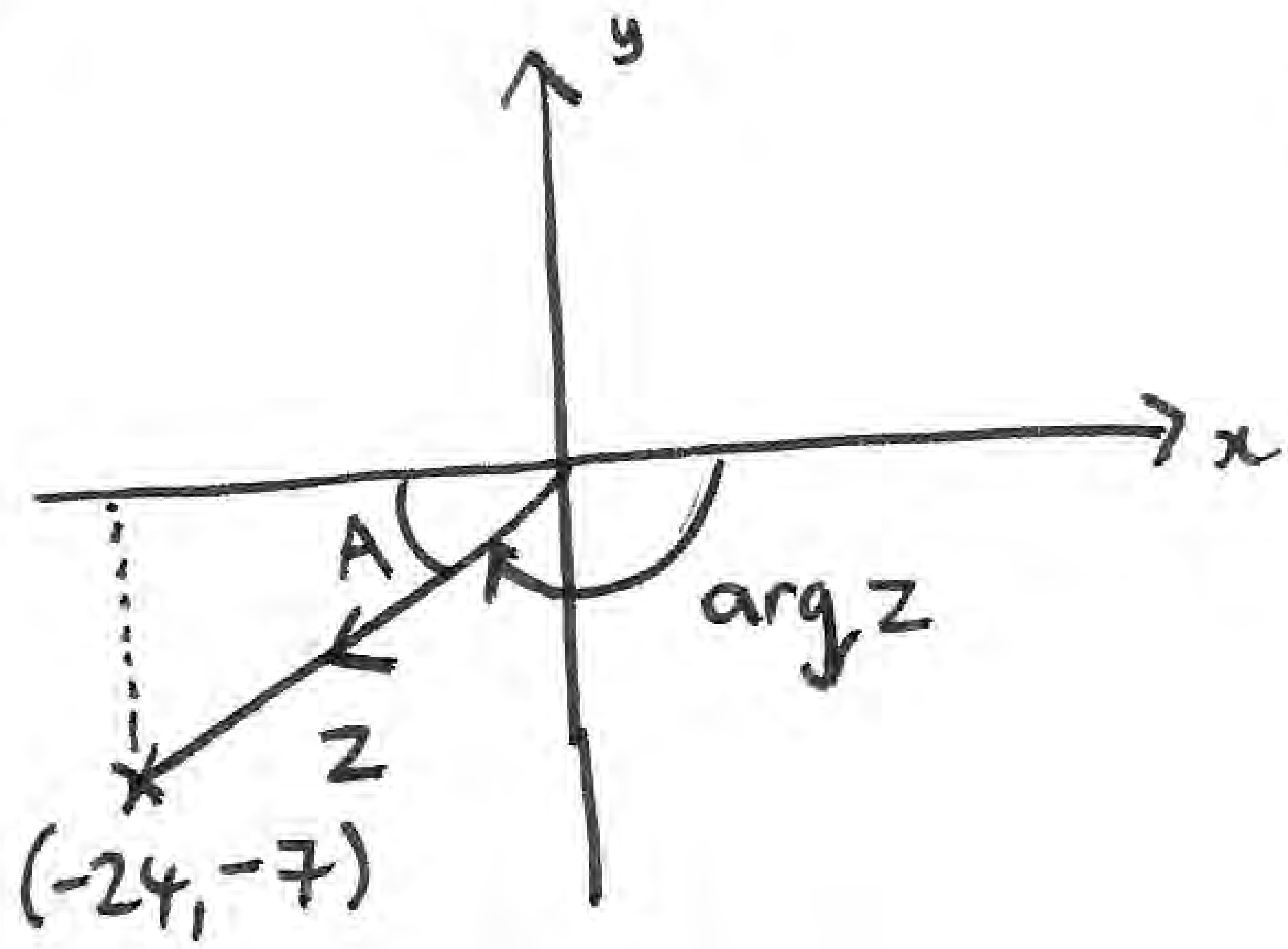
$$w = a + b\text{i}, \quad a \in \mathbb{R},\ b \in \mathbb{R}$$

Given also that $|w| = 4$ and $\arg w = \dfrac{5\pi}{6}$,

(c) find the values of a and b, **(3)**

(d) find the value of $|zw|$. **(3)**

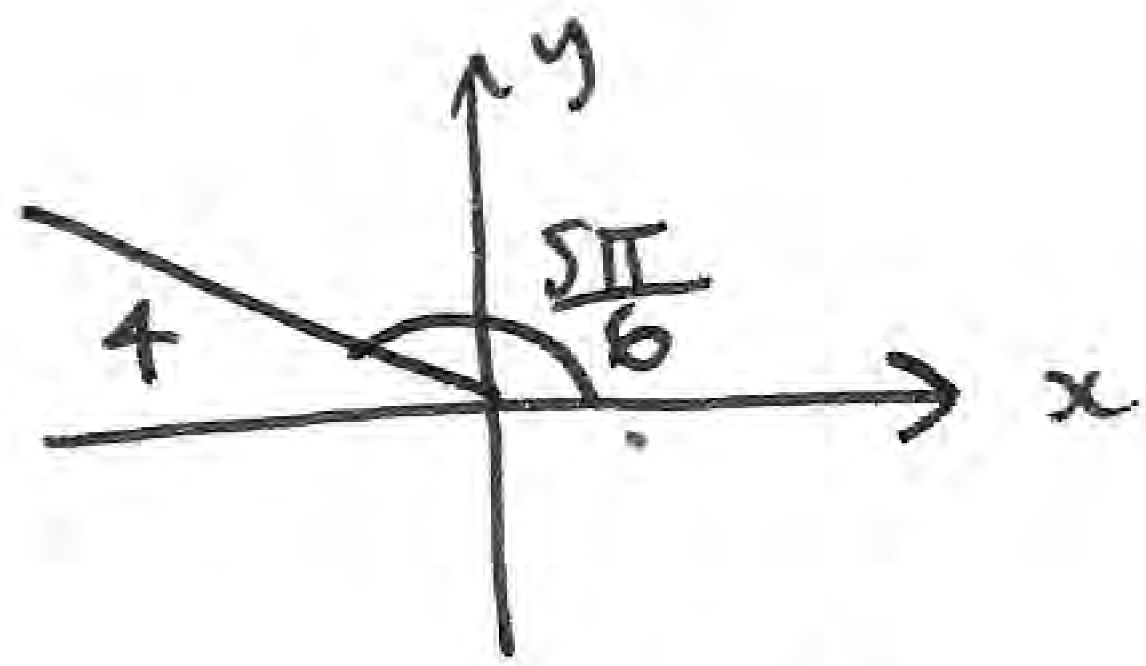
a)

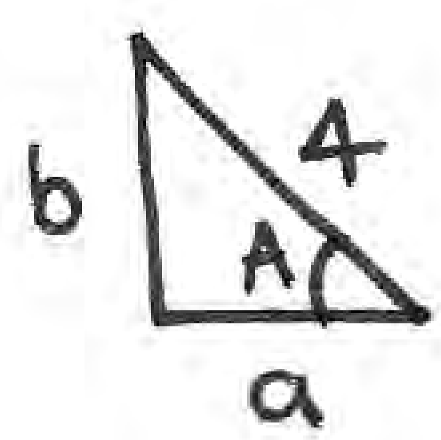


b) $\arg z = -\left(\pi - \tan^{-1}\left(\frac{7}{24}\right)\right)$

$\arg z = -2.86^c$

c)





$A = \frac{\pi}{6}$

$a = 4\cos\frac{\pi}{6} = 2\sqrt{3}$

$b = 4\sin\frac{\pi}{6} = 2$

$w = -2\sqrt{3} + 2i$

d) $|zw| = |z||w| = 25 \times 4 = \underline{100}$

8.

$$\mathbf{A} = \begin{pmatrix} 2 & -2 \\ -1 & 3 \end{pmatrix}$$

(a) Find det **A**. **(1)**

(b) Find $\mathbf{A}^{-1}$. **(2)**

The triangle R is transformed to the triangle S by the matrix **A**.
Given that the area of triangle S is 72 square units,

(c) find the area of triangle R. **(2)**

The triangle S has vertices at the points $(0, 4)$, $(8, 16)$ and $(12, 4)$.

(d) Find the coordinates of the vertices of R. **(4)**

a) $\det A = 2\times3 - (-2)(-1) = 6 - 2 = 4$

b) $A^{-1} = \frac{1}{4}\begin{pmatrix} 3 & 2 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} \frac{3}{4} & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{2} \end{pmatrix}$

c) $R \times \det A = S$

area $R = \frac{72}{4} = 18$

d) $\begin{pmatrix} \frac{3}{4} & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{2} \end{pmatrix}\begin{pmatrix} 0 & 8 & 12 \\ 4 & 16 & 4 \end{pmatrix} = \begin{pmatrix} 2 & 14 & 11 \\ 2 & 10 & 5 \end{pmatrix}$

$(2,2)$; $(14,10)$; $(11,5)$

9. A sequence of numbers $u_1, u_2, u_3, u_4, \ldots$ is defined by

$$u_{n+1} = 4u_n + 2, \quad u_1 = 2$$

Prove by induction that, for $n \in \mathbb{Z}^+$,

$$u_n = \frac{2}{3}\left(4^n - 1\right)$$

(5)

$U_1 = \frac{2}{3}(4^1 - 1) = \frac{2}{3} \times 3 = 2$ ✓

$U_2 = \frac{2}{3}(4^2 - 1) = \frac{2}{3} \times 15 = 10$

$U_2 = 4U_1 + 2 = 4 \times 2 + 2 = 10$ ✓

assume rule works for $n = k$

$U_k = \frac{2}{3}(4^k - 1)$

$U_{k+1} = \frac{2}{3}(4^{k+1} - 1)$

$U_{k+1} = 4\left[\frac{2}{3}(4^k - 1)\right] + 2$

$= \frac{8}{3}(4^k - 1) + 2 = \frac{2}{3}\left[4(4^k - 1) + 3\right]$

$= \frac{2}{3}\left[4^{k+1} - 4 + 3\right]$

$= \frac{2}{3}\left[4^{k+1} - 1\right]$ #.

rule works for $n=1, n=2$; if it works for $n=k$, it must work for $n=k+1$ so it is true for all $n \geq 1, n \in \mathbb{Z}^+$ by mathematical induction.

10. The point $P\left(6t, \frac{6}{t}\right), t \neq 0$, lies on the rectangular hyperbola H with equation $xy = 36$.

(a) Show that an equation for the tangent to H at P is

$$y = -\frac{1}{t^2}x + \frac{12}{t}$$

(5)

The tangent to H at the point A and the tangent to H at the point B meet at the point $(-9, 12)$.

(b) Find the coordinates of A and B.

(7)

a) $xy = 36 = c^2 \quad x = ct \quad y = \frac{c}{t}$

$\Rightarrow c = 6 \quad x = 6t \quad y = \frac{6}{t}$

$y = 36x^{-1}$

$\frac{dy}{dx} = -36x^{-2} = \frac{-36}{x^2} \quad x = 6t \quad m_t = \frac{-36}{36t^2} = \frac{-1}{t^2}$

$y - y_1 = m(x - x_1) \Rightarrow y - \frac{6}{t} = \frac{-1}{t^2}(x - 6t)$

$y - \frac{6}{t} = -\frac{1}{t^2}x + \frac{6}{t} \Rightarrow y = -\frac{1}{t^2}x + \frac{12}{t}$ #

b) $12 = \frac{-1}{t^2} \times -9 + \frac{12}{t} \Rightarrow 12 = \frac{9}{t^2} + \frac{12}{t} \quad \times t^2$

$12t^2 = 9 + 12t \Rightarrow 4t^2 - 4t - 3 = 0 \Rightarrow (2t+1)(2t-3)$

$t = \frac{-1}{2}, t = \frac{3}{2} \quad t = \frac{-1}{2} \quad \cancel{(6t, \frac{6}{t})} \ (-3, -12)$

$t = \frac{3}{2} \quad (9, 4)$